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ON THE NUMBER OF DISJOINT INCREASING PATHS IN THE ONE-SKELETON OF A CONVEX BODY LEADING TO A GIVEN EXPOSED FACE

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ABSTRACT

It is shown that if F is an n-dimensional exposed face of a d-dimensional convex body and if f is a linear functional whose maximum value on the body is attained over the whole of F then $n+1$ paths can be found in the one-skeleton of the body leading to F and disjoint except at their end-points. Further, such paths may be found having the property that along them, the value of f strictly increases. It is further shown that unless $n=0$ it may be impossible to find $n+2$ such paths.

1. Introduction

A programme of research, initiated by D. G. Larman and C. A. Rogers [1, 2], aims to extend results from the field of convex polytopes to more general results about convex bodies.

So far, attention has been focused on the properties of the one-skeleton of a convex body, the union of its extreme points and extreme edges. In particular, the connectedness of this structure has been studied.

A path in the one-skeleton of a convex body is defined as the image of the unit interval [0, 1] under a continuous injective mapping. This definition gives an analogue to the usual finite graph theoretic definition used for paths in the one-skeleton of a polytope.

A directed path in the one-skeleton of a convex body may be defined with respect to any non-constant linear functional f. A path S, the image of [0, 1] under the continuous injective mapping s, is said to be f-strictly increasing if $0 \leqq u_1 < u_2 \leqq 1$ implies $f(s(u_1)) < f(s(u_2))$. Such directed paths arise very naturally. In the case of polytopes, for example, they may be constructed using the simplex algorithm of linear programming.

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It has been shown [3] that the one-skeleton of a d-dimensional convex body K must contain d disjoint paths with the property that along each path f assumes values in a range arbitrarily close to its range on K.

In the same paper, a three-dimensional convex body K is constructed which is used to show that the disjoint path result given is, in a sense, best possible. The value of the underlying functional f assumes its maximum value on K over the whole of an exposed edge E. The structure of the one-skeleton of K is such that whenever three strictly increasing paths are given which lead to E, two of the paths must intersect in a point which is not a point of E.

A problem suggests itself: Suppose F is an n-dimensional exposed face of a d-dimensional convex body K. Suppose further that f is a non-constant linear functional on E^d whose maximum value on K is assumed on the whole of the face F. How many strictly increasing paths can be found in the one-skeleton of K, which lead to F and which are disjoint except at their endpoints?

In this paper a solution is given to this problem. It is shown that it is always possible to find $n+1$ such paths but, unless $n=0$, it may be impossible to find $n+2$. In the case $n=0$ Larman and Rogers have shown that it is always possible to find two such paths [2].

2. Increasing paths leading to a given exposed face

Before proving the existence of $n+1$ disjoint paths leading to the exposed face F, a result of Larman and Rogers must be quoted. The result concerns the measure of the directions of a certain set of line-segments in the boundary of a convex body. In their work they represent the direction of a line-segment by the end-points of the two unit vectors parallel to the line-segment.

LEMMA 1. *If H_1 and H_2 are two parallel support hyperplanes to a d-dimensional convex body K, meeting K in disjoint faces F_1 and F_2, then the set of directions of line-segments in* $\operatorname{bd} K \setminus \operatorname{relint} F_1 \setminus \operatorname{relint} F_2$ *has zero $(d-2)$-dimensional Hausdorff measure.*

PROOF. See [2].

THEOREM 1. *If K is a d-dimensional convex body and f is a non-constant linear functional on E^d whose maximum value on K is attained over the whole of an n-dimensional exposed face F, then there are $n+1$ paths in the one-skeleton of K, disjoint except at their end-points, along each of which f strictly increases to its maximum value on K.*

PROOF. This result is proved by induction on the dimension of K. The result is trivial in E^2. Proceeding with an inductive argument, assume the truth of the theorem for all dimensions less than d.

The case where $n = d - 1$ forms the main result of a previous work [3, theorem 1]. It may be assumed, then, that $n < d - 1$.

The hyperplane M given by

$$M = \left\{ \boldsymbol{x} : \boldsymbol{x} \in E^d,\ f(\boldsymbol{x}) = \max_{\boldsymbol{y} \in K} f(\boldsymbol{y}) \right\}$$

meets K in the face F.

By Lemma 1, since $\dim F < d - 1$, there must be some direction $\boldsymbol{p}$ which is parallel to M, yet not parallel to any line-segment in $\mathrm{bd}\, K$.

Denoting the orthogonal projection mapping in direction $\boldsymbol{p}$ by proj_p, $\mathrm{proj}_p(F)$ is an n-dimensional exposed face of the $(d-1)$-dimensional convex body $\mathrm{proj}_p(K)$.

By the inductive hypothesis, there must be paths $S_1, \cdots, S_{n+1}$ in the one-skeleton of $\mathrm{proj}_p(K)$, disjoint except at their end-points, along each of which f strictly increases to its maximum value on $\mathrm{proj}_p(K)$.

Inverse projection of these paths results in $n+1$ similar paths in the one-skeleton of K. This establishes the theorem.

3. Construction of counterexamples

In this section it is shown that the result given in Theorem 1 is in a sense best-possible. The following result is obtained:

THEOREM 2. *Suppose f is a non-constant linear functional on E^d and that n is an integer with $0 < n \leqq d - 1$. There is a d-dimensional convex body K with an n-dimensional exposed face F, over the whole of which the maximum value of f on K is attained, such that whenever $n + 2$ f-strictly increasing paths in the one-skeleton of K lead to F, then at least two of the paths intersect in a point which is not a point of F.*

The essential construction of the convex body K is carried out in Lemma 2. The necessarily complicated statement of the lemma conceals a very intuitive construction.

The convex body K is formed like a layer-cake. A sequence of polytopes are stacked one upon the other, the "layers" of the structure forming level sets for the linear functional f. The i-th layer of the structure is made to contain a large

n-dimensional face $F^{(i)}$, large in the sense that every edge of the face has length exceeding some positive number η, a number independent of i. The sequence of these large faces tends to the face F.

Consecutive faces $F^{(i)}$ and $F^{(i+1)}$ are constructed very close to one another, corresponding vertices joined by edges. The $n+1$ different sequences of such edges can be used to form $n+1$ paths in the one-skeleton of K leading to the face F. The lemma ensures that no other path can be found.

At the i-th level, vertices which are not in $F^{(i)}$ are grouped very close together, yet a long way ($>\eta - 3/2^i$) from corresponding vertices in the previous level. Any path using a sequence of such vertices would necessarily contain an infinite sequence of line-segments of equal length. This would contradict the continuity properties of paths.

Before proceeding with formal proof, certain notation is required: If f is a non-constant linear functional on E^d and α is a real number, then denote by $H(\alpha)$ the hyperplane given by

$$H(\alpha) = \{x : x \in E^d,\ f(x) = \alpha\}.$$

Denote by $H(\alpha)$ the closed half-space given by

$$H(\alpha) = \{x : x \in E^d,\ f(x) \leqq \alpha\}.$$

LEMMA 2. *Suppose f is a non-constant linear functional on E^d. Suppose n is an integer with $0 < n \leqq d-2$ and that η is a positive real number. Then there is a bounded sequence of d-polytopes $\{P^{(i)}\}_{i=1}^{\infty}$ such that:*

(i) *If $\alpha^{(i)} = \max_{x \in P^{(i)}} f(x)$ then the sequence $\{\alpha^{(i)}\}$ is strictly increasing.*

(ii) $P^{(i+1)} \cap H(\alpha^{(i)}) = P^{(i)}$, $i = 1, 2, \cdots$.

(iii) $\Delta^{(i)} = P^{(i)} \cap H(\alpha^{(i)}) = \operatorname{conv}\{x_0^{(i)}, \cdots, x_{d-1}^{(i)}\}$ *is a $d-1$-dimensional simplex,*

(iv) $F^{(i)} = \operatorname{conv}\{x_0^{(i)}, \ldots, x_n^{(i)}\}$ *is an n-dimensional simplicial face of $\Delta^{(i)}$ with* $\min_{0 \leqq j < k \leqq n} |x_j^{(i)} - x_k^{(i)}| > \eta$, $i = 1, 2, \cdots$.

(v) $\max_{0 \leqq j \leqq n} |x_j^{(i)} - x_j^{(i+1)}| < 1/2^i$, $i = 1, 2, \cdots$.

(vi) *The point $x_j^{(i)}$ is joined to the point $x_j^{(i+1)}$ by a single edge of the polytope* $P^{(i+1)}$, $0 \leqq j \leqq n$, $i = 1, 2, \cdots$.

(vii) $\min_{n < j < d} |x_j^{(i)} - x_{i \bmod 2}^{(i)}| < 1/2^i$, $i = 1, 2, \cdots$.

(viii) *If $n+2$ disjoint f-strictly increasing paths in the one-skeleton of $P^{(i+1)}$ lead from $\Delta^{(i)}$ to $\Delta^{(i+1)}$ then one must contain a line-segment of length exceeding* $\eta - 3/2^i$, $i = 1, 2, \cdots$.

PROOF. A suitable sequence is constructed inductively. The construction of the first two members of such a sequence is trivial. Assume, then, that a finite sequence of m polytopes can be found having properties (i) to (viii). It will be shown that a polytope $P^{(m+1)}$ can be constructed so that the enlarged sequence also satisfies the required conditions.

Let $\boldsymbol{d}$ be the apex of a shallow pyramid to $P^{(m)}$ over the face $\Delta^{(m)}$. The polytope $P^{(m+1)}$ is constructed by suitably truncating the polytope $\mathrm{conv}(P^{(m)}\cup\{\boldsymbol{d}\})$. The choice of $\boldsymbol{d}$ can easily be made so that a bounded sequence of polytopes results.

Let $\alpha^{(m+1)}$ be chosen greater than $\alpha^{(m)}$, so that with $\boldsymbol{x}_j^{(m+1)}$ defined for $0 \leqq j \leqq n$ by $\boldsymbol{x}_i^{(m+1)} = \mathrm{conv}\{\boldsymbol{x}_i^{(m)}, d\} \cap H(\alpha^{(m+1)})$, the inequalities of conditions (iv) and (v) are satisfied. Elementary geometry can be used to show that this is always possible.

Let T denote the face of $\mathrm{conv}(P^{(m)}\cup\{\boldsymbol{d}\}) \cap H(\alpha^{(m+1)})$ lying in the hyperplane $H(\alpha^{(m+1)})$.

The points $\boldsymbol{x}_0^{(m+1)}, \cdots, \boldsymbol{x}_n^{(m+1)}$, defined above, are all vertices of T. Let the remaining vertices of T be $\boldsymbol{y}_{n+1}^{(m+1)}, \cdots, \boldsymbol{y}_{d-1}^{(m+1)}$, ordered naturally so that the vertex $\boldsymbol{y}_i^{(m+1)}$ is joined to $\boldsymbol{x}_i^{(m+1)}$ by a single edge of the polytope $\mathrm{conv}(P^{(m)}\cup\{\boldsymbol{d}\}) \cap H(\alpha^{(m+1)})$.

For $n+1 \leqq j < d$, let $\boldsymbol{x}_j^{(m+1)}$ be a point on the edge $[\boldsymbol{y}_j^{(m+1)}, \boldsymbol{x}_{m \bmod n}^{(m+1)}]$, an edge of T, chosen so that $|\boldsymbol{x}_j^{(m+1)} - \boldsymbol{x}_{m \bmod 2}^{(m+1)}| < 1/2^{m+1}$ and so that the convex set $\Delta^{(m+1)} = \mathrm{conv}(\boldsymbol{x}_0^{(m+1)}, \cdots, \boldsymbol{x}_{d-1}^{(m+1)})$ is a d-1-dimensional simplex.

The polytope $P^{(m+1)}$ is defined to be the convex hull of $P^{(m)}$ and $\Delta^{(m+1)}$. Conditions (i) to (vii) apply to the sequence of polytopes $P^{(1)}, \cdots, P^{(m+1)}$ by construction. Only condition (viii) requires proof.

It is useful to adopt the following nomenclature: A vertex of $P^{(m+1)}$ will be called an F-vertex if it is contained in some face $F^{(j)}$ with $1 \leqq j \leqq m+1$.

Suppose $S_0, \cdots, S_{n+1}$ are $n+2$ disjoint strictly increasing paths in the one-skeleton of $P^{(m+1)}$ joining $\Delta^{(m)}$ to $\Delta^{(m+1)}$. Suppose further that none of these paths contains a line-segment of length exceeding $\eta - 3/2^m$.

The distance between any non-F-vertex in $\Delta^{(m)}$ and any non-F-vertex in $\Delta^{(m+1)}$ exceeds $\eta - 3/2^m$ since if $\boldsymbol{x}_j^{(m)}$ and $\boldsymbol{x}_k^{(m+1)}$ are two such points then

$$
\begin{aligned}
|\boldsymbol{x}_k^{(m+1)} - \boldsymbol{x}_j^{(m)}| &\geqq |\boldsymbol{x}_{m+1 \bmod 2}^{(m+1)} - \boldsymbol{x}_{m \bmod 2}^{(m)}| - |\boldsymbol{x}_{m+1 \bmod 2}^{(m+1)} - \boldsymbol{x}_k^{(m+1)}| - |\boldsymbol{x}_j^{(m)} - \boldsymbol{x}_{m \bmod 2}^{(m)}| \\
&> |\boldsymbol{x}_{m+1 \bmod 2}^{(m+1)} - \boldsymbol{x}_{m \bmod 2}^{(m)}| - 1/2^{m+1} - 1/2^m \qquad \text{by condition (vii)} \\
&> |\boldsymbol{x}_{m+1 \bmod 2}^{(m)} - \boldsymbol{x}_{m \bmod 2}^{(m)}| - |\boldsymbol{x}_{m+1 \bmod 2}^{(m+1)} - \boldsymbol{x}_{m+1 \bmod 2}^{(m)}| - 3/2^{m+1} \\
&> |\boldsymbol{x}_{m+1 \bmod 2}^{(m)} - \boldsymbol{x}_{m \bmod 2}^{(m)}| - 1/2^m - 3/2^{m+1} \qquad \text{by condition (v)} \\
&> \eta - 3/2^m \qquad \text{by condition (iv).}
\end{aligned}
$$

A similar inequality-chasing argument can be used to show that if $\boldsymbol{x}_k^{(m+1)}$ is a non-

F-vertex in $\Delta^{(m+1)}$, then it can only be joined to one vertex in $\Delta^{(m)}$ by an edge with length less than $\eta - 3/2^m$. The one vertex is $x^{(m)}_{m+1 \bmod 2}$. Again, non-F-vertices in $\Delta^{(m)}$ can only be joined to $x^{(m+1)}$ in $\Delta^{(m+1)}$ using edges of length less than $\eta - 3/2^m$.

One of the paths $S_0, \cdots, S_{n+1}$ must lead from a non-F-vertex in $\Delta^{(m)}$. Since none of the paths contains a line-segment of length exceeding $\eta - 3/2^m$, the path must lead to the vertex $x^{(m+1)}_{m \bmod 2}$. No other non-F-vertices in $\Delta^{(m)}$ can be contained in any of the paths.

Similarly, one of the paths must lead to a non-F-vertex in $\Delta^{(m+1)}$ and must have emanated from the vertex $x^{(m)}_{m+1 \bmod 2}$. No other non-F-vertices in $\Delta^{(m+1)}$ can be a point in any of the paths.

A contradiction is now apparent. There remain n disjoint paths between $\Delta^{(m)}$ and $\Delta^{(m+1)}$. These paths can only contain F-vertices and may not pass through the points $x^{(m)}_{m+1 \bmod 2}$ or $x^{(m+1)}_{m \bmod 2}$. One of the paths must join two F-vertices which have a different index, $x^{(m)}_j$ and $x^{(m+1)}_k$. Again, an inequality-chasing argument using conditions (iv) and (v) can show that $|x^{(m)}_j - x^{(m+1)}_k| > \eta - 3/2^m$. This contradiction establishes condition (viii) for the sequence of polytopes $P^{(1)}, \cdots, P^{(m+1)}$. The lemma now follows by induction.

A further lemma is required for the proof of Theorem 2.

LEMMA 3. *It is impossible for a path in the one-skeleton of a convex body to contain, as sub-arcs, a sequence of line-segments which are disjoint and which all have length exceeding some positive number η^*.*

PROOF. This result follows from a straight-forward continuity argument using the fact that a path in the one-skeleton is a continuous image of $[0, 1]$.

PROOF OF THEOREM 2. Under the notation of Lemma 2, define the convex body K by

$$K = \mathrm{cl} \bigcup_{i=1}^{\infty} P^{(i)}.$$

In view of condition (iv) of Lemma 2, it is clear that the functional f assumes its maximum value on K over the whole of a face F which must have dimension at least n. (The face F must contain a set which is the limit of the sequence of n-polytopes $\{F^{(i)}\}_{i=1}^{\infty}$. This set clearly has dimension n.)

It is impossible to find $n+2$ paths in the one-skeleton of K which lead to F, yet which are disjoint outside F. If such paths did exist then, by condition (viii) of Lemma 2 (suitably taken to the limit), one of the paths would contain, as subarcs, a sequence of disjoint line-segments of length exceeding $\eta^* = \eta/2$. By Lemma 3, this is impossible.

In view of Theorem 1, the dimension of F must be exactly n.

REMARK. In the construction carried out in Lemma 2, the assumption that the paths between successive simplicial sections $\Delta^{(i)}$ and $\Delta^{(i+1)}$ should be strictly increasing is not necessary to the proof. Thus there is the rather surprising result that it is impossible to find any $n + 2$ disjoint paths leading to the n-dimensional face F.

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